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Technical Notes

New Orthotropic, Two-Dimensional, Transient Heat-Flux/Temperature Integral Relationship for Half-Space Diffusion

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Nomenclature

C = heat capacity, $\mathrm{kJ/(kg°C)}$
k = thermal conductivity, $\mathrm{W/(m°C)}$
k_{xx} = thermal conductivity coefficient, $\mathrm{W/(m°C)}$
k_{xy} = thermal conductivity coefficient, $\mathrm{W/(m°C)}$
k_{yx} = thermal conductivity coefficient, $\mathrm{W/(m°C)}$
k_{yy} = thermal conductivity coefficient, $\mathrm{W/(m°C)}$
q''_o = dimensional surface heat flux, $\mathrm{W/m^2}$
q''_x = dimensional heat flux in the x direction, $\mathrm{W/m^2}$
q''_y = dimensional heat flux in the y direction, $\mathrm{W/m^2}$
T = temperature, °C
t = time, s
t_o = dummy time variable, s
t' = dummy time variable, s
x = spatial variable, m
y = spatial variable, m
$y_{\max}$ = rescaled y variable, m
y_o = dummy spatial variable, m
y' = dummy spatial variable, m
α = thermal diffusivity, $k_{xx}/(\rho C)$, $\mathrm{m^2/s}$
ϵ_1 = k_{yy}/k_{xx}
η = arbitrary x location, m
ρ = density, $\mathrm{kg/m^3}$

I. Introduction

THIS Note derives a new integral relationship between heat flux and temperature for a transient two-dimensional heat-conducting Cartesian half-space [$x > 0$ and $y \in (-\infty, \infty)$] having an orthotropic thermal conductivity. A unified mathematical treatment is proposed based on operational and transform methods and singular integral-equation regularization. The diffusive operator provides sufficient insight into choosing the integral-equation regularization. The resulting heat-flux–temperature (q''–T) relationship has practical implications for experimental studies. The local heat flux perpendicular to the front surface at fixed depth can be estimated through a linear array of embedded temperature sensors parallel to the surface. Knowledge of the surface boundary condition is not required, since it has analytically been removed through the proposed mathematical treatment. The ill-posed nature of diffusion is observed, since knowledge of the heating/cooling rate (°C/s) is explicitly required in the integrand of the new relationship.

A working relationship between measured transient temperatures and heat fluxes [1] is critical to engineers for developing and evaluating new aerospace materials such as those required in thermal protection systems. Heat shields must be highly effective while remaining relatively lightweight. Computer simulation alone based on a proposed model can lead to an erroneous estimate of the new material effectiveness. Laboratory tests may significantly contrast from the resulting simulations. Thus, rigorous testing and evaluation are necessary when embedded sensors can be used to extract useful information on temperature and energy content. This in turn can be useful for assessing the accuracy of the computer simulations and for potentially locating defects in the theoretical–computational model.

Heat flux is normally estimated through a temperature gradient involving the appropriate coordinate direction. For isotropic materials, Fourier's law is used for estimating heat fluxes. Heat-flux gauges are normally designed and fabricated based on this observed law. However, it is possible to acquire the heat flux in an alternate manner using the time domain for transient studies. This is possible if the general law (first law of thermodynamics) and constitutive equation for heat flux (Fourier's law) are combined to obtain a heat equation in the scalar variable of temperature. Frankel et al. [2–5] have developed a unified mathematical treatment involving isotropic materials in one- and two-dimensional planar half-spaces. Kulish et al. [6,7] developed T–q'' one-dimensional relationships based on Laplace transforms. In these investigations, the heat flux was provided and the temperature was acquired. Our proposed treatment leads to a new heat-flux–temperature integral relationship that does not require spatial gradients. Hence, only time histories of temperature are required to obtain the heat flux. Determining heat flux from temperature is ill-posed, whereas determining temperature from heat flux is well-posed [2]. This offers some hope for estimating embedded heat fluxes using temperature sensors through a time-domain analysis.

This Note presents a new two-dimensional, orthotropic, transient, half-space, heat-flux–temperature integral relationship. This is often a practical geometry for material testing and can be used in studies in which the thermal signal has yet to fully penetrate the finite body. The orthotropic case reduces to the known isotropic results [3] in the limit as the thermal conductivity coefficients [8] become equal. It is our intent to derive the two-dimensional transient heat-flux–temperature integral relationship in the half-space having an orthotropic thermal conductivity subject to the trivial (uniform) initial condition. The resulting relationship is

$$q''_x(\eta, y, t) = \frac{k_{xx}}{2\pi\alpha\sqrt{\epsilon_1}} \int_{t_o=0}^{t} \int_{y_o=-\infty}^{\infty} \left(\frac{\partial T}{\partial t_o}(\eta, y_o, t_o) + \frac{T(\eta, y_o, t_o)}{2(t-t_o)} \left(1 - \frac{(y-y_o)^2}{2\alpha\epsilon_1(t-t_o)} \right) \right) \frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{t-t_o} \, \mathrm{d}y_o \, \mathrm{d}t_o$$
$$y \in (-\infty, \infty), \qquad (x, t) \geq 0 \tag{1}$$

where $x = \eta$ defines the x position where the sensor array lies. Here, $T(x, y, t)$ is the temperature, $q''_x(x, y, t)$ is the heat flux in the x direction, (x, y) are the coordinate axes such that $x \geq 0$ and $y \in (-\infty, \infty)$, and t is time. The parameter $\epsilon_1 = k_{yy}/k_{xx}$, where k_{xx} is the conductivity coefficient corresponding to the x direction and k_{yy} is the conductivity coefficient corresponding to the y direction.

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The thermal diffusivity α is defined as $\alpha = k_{xx}/(\rho C)$, where ρ is the density and C is the heat capacity. When $k_{xx} = k_{yy} = k$, Eq. (1) reduces to the isotropic result given in [3]. Figure 1 displays a geometric understanding for the implementation of Eq. (1). Per Fig. 1, sensors located at fixed $x = \eta$ collect transient temperature data. These data are used as input to the integrand described in Eq. (1). The resulting double integral must be numerically integrated [3] by quadratures. A numerical implementation for the isotropic case has been provided in [3].

Observe that Eq. (1) does not possess a spatial gradient in order to obtain the heat flux, since this expression encompasses both the first law of thermodynamics and the constitutive equations (Fourier's law in each coordinate direction). Additionally, this mathematically convenient geometry can provide insight into the ill-posed nature of the inverse problem. This Note describes an operational methodology representing a unified theoretical treatment. Section II provides the mathematical formulation and resulting exact solutions for describing orthotropic heat conduction in the half-space. The exact solution is obtained from conventional transform methods [8,9] and reduces to the isotropic solution through a simple transform of the y variable or by letting $\epsilon_1 = 1$ ($k_{xx} = k_{yy} = k$). Section III describes the regularization procedure leading to the novel heat-flux–temperature integral relationship given in Eq. (1). Finally, Section IV provides a summary statement.

II. Formulation and Exact Solution

Consider the anisotropic transient heat equation in the two-dimensional semi-infinite planar region given by [8]

$$\rho C \frac{\partial T}{\partial t}(x,y,t) = k_{xx}\frac{\partial^2 T}{\partial x^2}(x,y,t) + k_{yy}\frac{\partial^2 T}{\partial y^2}(x,y,t) + 2k_{xy}\frac{\partial^2 T}{\partial x \partial y}(x,y,t), \quad (x,t) \geq 0, \quad y \in (-\infty,\infty) \tag{2a}$$

subject to the trivial initial condition

$$T(x,y,0) = 0, \qquad x \geq 0, \qquad y \in (-\infty,\infty) \tag{2b}$$

The heat fluxes given by

$$q''_x(x,y,t) = -k_{xx}\frac{\partial T}{\partial x}(x,y,t) - k_{xy}\frac{\partial T}{\partial y}(x,y,t) \tag{3a}$$

$$q''_y(x,y,t) = -k_{yy}\frac{\partial T}{\partial y}(x,y,t) - k_{yx}\frac{\partial T}{\partial x}(x,y,t) \tag{3b}$$

have been implicitly used in arriving at Eq. (2a), since

$$\rho C \frac{\partial T}{\partial t} = -\frac{\partial q''_x}{\partial x} - \frac{\partial q''_y}{\partial y}$$

From Onsager's reciprocity relationship [8], the conductivity coefficient $k_{xy} = k_{yx}$. The orthotropic case is defined as $k_{xy} = k_{yx} = 0$. Thus, Eq. (2a) reduces to

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}(x,y,t) = \frac{\partial^2 T}{\partial x^2}(x,y,t) + \epsilon_1 \frac{\partial^2 T}{\partial y^2}(x,y,t) \qquad (x,t) \geq 0, \qquad y \in (-\infty,\infty) \tag{4}$$

subject to the initial condition described in Eq. (2b) and where α and ϵ_1 are previously defined. The imposed trivial (or constant using a linear transformation) initial condition suggests that significant support exists as $x \to \infty$: namely,

$$\frac{\partial^n T}{\partial x^n}(\infty,y,t) = 0$$

for $n = 0, 1, \ldots$ and as $y \to \pm\infty$: namely,

$$\frac{\partial^n T}{\partial y^n}(x,\pm\infty,t) = 0$$

An analytical investigation is proposed using classical operational methods involving Fourier transforms [8,9], integral-equation regularization as associated with singular integral-equation theory, and a series of mathematical observations. The application of Fourier transforms (Fourier cosine transform in the x direction and exponential Fourier transform in the y direction) applied to the heat equation shown in Eq. (4) subject to the initial condition displayed in Eq. (2b) with the associated support at infinity ($x \to \infty$ and $y \to \pm\infty$) leads to the temperature solution

$$T(x,y,t) = \frac{1}{2k_{xx}\pi\sqrt{\epsilon_1}}\int_{t_o=0}^{t}\int_{y_o=-\infty}^{\infty} q''_x(0,y_o,t_o)\frac{e^{-\frac{x^2\epsilon_1+(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{t-t_o}\,\mathrm{d}y_o\,\mathrm{d}t_o \qquad y \in (-\infty,\infty) \qquad (x,t) \geq 0 \tag{5}$$

when provided a surface heat-flux boundary condition. The heat flux $q''_x(x,y,t)$ is obtained using Eq. (3a) when $k_{xy} = 0$ and using Eq. (5) to obtain

$$q''_x(x,y,t) = \frac{x}{4\pi\alpha\sqrt{\epsilon_1}}\int_{t_o=0}^{t}\int_{y_o=-\infty}^{\infty} q''_x(0,y_o,t_o)\frac{e^{-\frac{x^2\epsilon_1+(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{(t-t_o)^2}\,\mathrm{d}y_o\,\mathrm{d}t_o \qquad y \in (-\infty,\infty) \qquad (x,t) \geq 0 \tag{6}$$

Equations (5) and (6) reduce to the known isotropic solutions as $\epsilon_1 \to 1$. Additionally, with the change of independent variable in y [let $y_{\text{new}} = y/\sqrt{\epsilon_1}$ in Eq. (4)], the orthotropic equation reduces to the well-known isotropic equation. It should be noted that one could develop the exact solution based on the surface temperature, since the choice of boundary condition is arbitrary, to arrive at the integral relationship between temperature and heat flux [5].

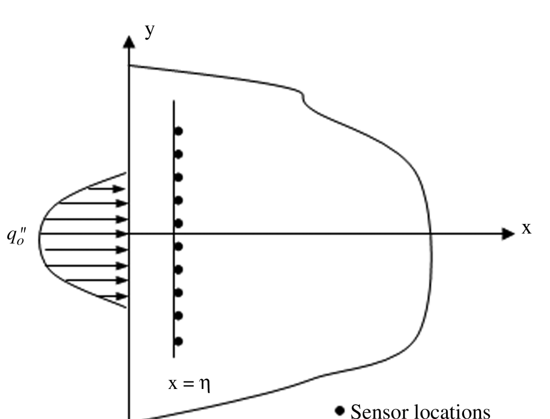

Fig. 1 Two-dimensional geometry indicating sensor strategy that corresponds to implementing Eq. (1).

III. Integral Relationship Regularization

Notice that the surface heat flux $q''_x(0,y,t)$ has not yet been specified. A mathematical procedure is proposed to eliminate the multidimensional integral [of Eqs. (5) and (6)] containing the surface-normal heat flux. This procedure contains four major steps beyond the development of the exact solution previously described in order to acquire a mathematical relationship between the heat flux in the x direction at arbitrary x position in terms of y-temperature-based data. This mathematical process requires the analytic removal of the surface heat-flux condition displayed in Eq. (5) or Eq. (6). These steps involve 1) operational regularization to assist in isolating and then later eliminating the surface heat-flux condition; 2) applying two spatial derivatives associated with the remaining independent variable (i.e., nonintegrated variables); 3) using the heat equation in the temperature variable to transfer the x derivatives into the y, t derivatives that are later integrated per the defined regions of integration; and 4) eliminating the surface boundary condition integral through identification leading to the desired heat-flux relationship given in Eq. (1). Operational regularization is most notably applied in Abel integral equations [10] associated with one independent variable as a mathematical device for inverting a weakly singular, first-kind equation in order to obtain the exact solution. It is also used in removing weak singularities through the iterated kernel [10]. This study requires generalization of the regularization concept, owing to the multidimensional nature of the proposed physical problem as the first step in the process of developing an integral relationship between heat flux and temperature.

A unified treatment is now proposed for such diffusive investigations. Two preliminary observations are required in order to derive this new relationship for $q_x''(x, y, t)$. First, the exhibited kernel in Eq. (5) is viewable as the product of two one-dimensional kernels: namely,

$$\frac{e^{-\frac{(x^2\epsilon_1+(y-y_o)^2)}{4\alpha\epsilon_1(t-t_o)}}}{(t-t_o)} = \frac{e^{-\frac{x^2}{4\alpha(t-t_o)}}}{\sqrt{t-t_o}}\frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{\sqrt{t-t_o}} \tag{7}$$

Second, a novel regularization must be introduced. From our previous one- and two-dimensional isotropic studies [2–5], the regularizing operator is evident from Eq. (7). Key to accomplishing this operational regularization is the observation that

$$\frac{1}{2\sqrt{\alpha\epsilon_1\pi}}\int_{y_o=-\infty}^{\infty}\frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{\sqrt{t-t_o}}\,\mathrm{d}y_o = 1 \tag{8}$$

This integral permits the analytic reduction in dimensionality and hence is suggestive of the proper kernel for operational regularization.

Replacing (y, t) by (y', t') in Eq. (5) and operating on this result with

$$\int_{t'=0}^{t}\int_{y'=-\infty}^{\infty}\frac{1}{\sqrt{t-t'}}\frac{e^{-\frac{(y-y')^2}{4\alpha\epsilon_1(t-t')}}}{\sqrt{t-t'}}\cdot\mathrm{d}y'\,\mathrm{d}t'$$

yields

$$\begin{aligned}\int_{t'=0}^{t}\int_{y'=-\infty}^{\infty}T(x,y',t')\frac{e^{-\frac{(y-y')^2}{4\alpha\epsilon_1(t-t')}}}{t-t'}\,\mathrm{d}y'\,\mathrm{d}t' &= \frac{1}{2\pi k_{xx}\sqrt{\epsilon_1}}\int_{t'=0}^{t}\int_{y'=-\infty}^{\infty}\frac{e^{-\frac{(y-y')^2}{4\alpha\epsilon_1(t-t')}}}{t-t'} \\ &\times\int_{t_o=0}^{t'}\int_{y_o=-\infty}^{\infty}q_{x_o}''(0,y_o,t_o)\frac{e^{-\frac{x^2\epsilon_1+(y'-y_o)^2}{4\alpha\epsilon_1(t'-t_o)}}}{t'-t_o}\,\mathrm{d}y_o\,\mathrm{d}t_o\,\mathrm{d}y'\,\mathrm{d}t' \\ y\in(-\infty,\infty),\qquad (x,t)\geq 0\end{aligned} \tag{9}$$

Interchanging orders of integration on the right-hand side (RHS) of Eq. (9) and analytically evaluating the resulting two inner integrals yields

$$\begin{aligned}\int_{t_o=0}^{t}\int_{y_o=-\infty}^{\infty}T(x,y_o,t_o)\frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{t-t_o}\,\mathrm{d}y_o\,\mathrm{d}t_o &= \frac{\sqrt{\alpha\pi}}{k_{xx}}\int_{t_o=0}^{t}\int_{y_o=-\infty}^{\infty}q_{x_o}''(0,y_o,t_o) \\ &\times\frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{\sqrt{t-t_o}}\operatorname{erfc}\sqrt{\frac{x^2}{4\alpha(t-t_o)}}\,\mathrm{d}y_o\,\mathrm{d}t_o \\ y\in(-\infty,\infty),\qquad (x,t)\geq 0\end{aligned} \tag{10a}$$

where

$$\int_{y'=-\infty}^{\infty}e^{-\frac{(y-y')^2}{4\alpha\epsilon_1(t-t')}}e^{-\frac{(y'-y_o)^2}{4\alpha\epsilon_1(t'-t_o)}}\,\mathrm{d}y' = 2\sqrt{\alpha\epsilon_1\pi}\sqrt{\frac{(t-t')(t'-t_o)}{t-t_o}}e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}} \tag{10b}$$

and with

$$\int_{t'=t_o}^{t}\frac{e^{-\frac{x^2}{4\alpha(t'-t_o)}}}{\sqrt{(t-t')(t'-t_o)}}\,\mathrm{d}t' = \pi\operatorname{erfc}\sqrt{\frac{x^2}{4\alpha(t-t_o)}},\qquad t-t_o\geq 0 \tag{10c}$$

and where $\operatorname{erfc}(u)$ is the complementary error function with real argument u. The dummy variables of integration on the left-hand side of Eq. (10a) have been changed for mere cosmetics.

Let us now express Eq. (4) as

$$\begin{aligned}\frac{1}{\alpha}\frac{\partial T}{\partial t_o}(x,y_o,t_o) &= \frac{\partial^2 T}{\partial x^2}(x,y_o,t_o)+\epsilon_1\frac{\partial^2 T}{\partial y_o^2}(x,y_o,t_o) \\ y_o\in(-\infty,\infty),\qquad (x,t_o)\geq 0\end{aligned} \tag{11}$$

Next, we operate on Eq. (10a) with $\partial^2/\partial x^2$ and then replace $(\partial^2 T/\partial x^2)(x, y_o, t_o)$ with Eq. (11) to obtain

$$\begin{aligned}\int_{t_o=0}^{t}\int_{y_o=-\infty}^{\infty}\left(\frac{1}{\alpha}\frac{\partial T}{\partial t_o}(x,y_o,t_o) - \epsilon_1\frac{\partial^2 T}{\partial y_o^2}(x,y_o,t_o)\right)\frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{t-t_o}\,\mathrm{d}y_o\,\mathrm{d}t_o &= \frac{x}{2\alpha k_{xx}}\int_{t_o=0}^{t}\int_{y_o=-\infty}^{\infty}q_{x_o}''(0,y_o,t_o)\frac{e^{-\frac{x^2\epsilon_1+(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{(t-t_o)^2}\,\mathrm{d}y_o\,\mathrm{d}t_o \\ y\in(-\infty,\infty),\qquad (x,t)\geq 0\end{aligned} \tag{12a}$$

since

$$\frac{\partial^2}{\partial x^2}\left(\operatorname{erfc}\sqrt{\frac{x^2}{4\alpha(t-t_o)}}\right) = \frac{x}{2\alpha\sqrt{\alpha\pi}}\frac{e^{-\frac{x^2}{4\alpha(t-t_o)}}}{(t-t_o)^{3/2}} \tag{12b}$$

Observe that the RHS of Eq. (12a) is now recognizable with the aid of Eq. (6) as the heat flux in the x direction at arbitrary x. That is, by multiplying Eq. (6) by $2\pi\sqrt{\epsilon_1}/k_{xx}$ we can eliminate the unwanted integral involving the surface heat flux displayed in the RHS of Eq. (12a). The spatial derivatives with respect to y_o on the temperature variable are removed through integration by parts and using the previously noted support at $y = \pm\infty$. Performing the integration by parts twice as previously noted and making use of Eq. (6) as previously described yields

$$q''_x(x,y,t) = \frac{k_{xx}}{2\pi\alpha\sqrt{\epsilon_1}} \int_{t_o=0}^{t} \int_{y_o=-\infty}^{\infty} \left(\frac{\partial T}{\partial t_o}(x, y_o, t_o) + \frac{T(x,y_o,t_o)}{2(t-t_o)} \left(1 - \frac{(y-y_o)^2}{2\alpha\epsilon_1(t-t_o)} \right) \right) \frac{e^{-\frac{(y-y_o)^2}{4\alpha\epsilon_1(t-t_o)}}}{t-t_o} \, \mathrm{d}y_o \, \mathrm{d}t_o$$

$$y \in (-\infty, \infty), \qquad (x,t) \geq 0 \tag{13}$$

Equation (13) reduces to the previously studied isotropic case [3] when $\epsilon_1 = 1$. This relationship suggests that the heat flux $q''_x(x,y,t)$ can be determined by installing a linear array of temperature sensors. The time histories of temperature at each probe site, in accordance with Fig. 1, is carefully interrogated for use in Eq. (13). The two-dimensional isotropic case ($\epsilon_1 = 1$) has been numerically presented in [3]. Equation (13) should have significant impact for the investigation of materials for future aerospace heat transfer studies involving material survivability.

IV. Conclusions

Equation (13) yields a novel integral relationship for the heat flux $q''_x(x,y,t)$ at any location in the semi-infinite geometry given the temperature histories at arbitrary $x = \eta$ per Fig. 1 along the y line for all time $t \geq 0$. The mathematical process is based on integral-equation regularization, as associated with singular integral-equation theory, and the observed kernel construction. The proposed observations offer new insight into experimental measurements for acquiring the heat flux based on embedded temperature sensors in an orthotropic material. Finally, Eq. (13) suggests that the direct measurement or proper digital filtering of the heating/cooling rate $(\partial T/\partial t)(x,y,t)$ may reduce or remove the ill-posed nature of differentiation [2–5].

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